DIFFERENT FORMS OF GASSMANN EQUATION AND IMPLICATIONS FOR THE ESTIMATION OF ROCK PROPERTIES

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Summary

Three new equivalent forms of Gassmann equation are presented that are useful when the unknown parameters are the Biot-Willis coefficient, the dry bulk modulus, and/or the grain matrix bulk modulus. We apply these equations to several sets of laboratory measurements to determine the profiles of grain matrix bulk modulus and Biot-Willis coefficient as functions of applied pressure, and perform a Monte Carlo simulation to examine the effect of uncertainty and/or measurement errors on the calculated grain matrix bulk modulus and Biot-Willis coefficient. The results show that the calculated grain matrix bulk modulus is relatively constant with applied differential pressure (up to 50MPa). However, it is very sensitive to the uncertainty of dry and saturated bulk modulus values. Thus, the presented new forms of Gassmann equation can be used to effectively quantify the uncertainty of dry and saturated bulk modulus (and subsequently, the seismic velocities) in fluid identification, fluid substitution, or reservoir monitoring applications.

Key words: *Gassmann equation, bulk moduli, Biot-Willis coefficient, sensitivity analysis.*

1. Introduction

The Gassmann equations [1] have been used extensively in the oil and gas industry for fluid identification and reservoir monitoring applications, despite its various assumptions [2, 3]. The first Gassmann equation provides the relationship between the saturated bulk modulus of a rock and its dry frame bulk modulus, porosity, bulk modulus of the mineral matrix, and bulk modulus of the pore-filling fluid. Whereas, the second Gassmann equation simply states that the shear modulus of the rock is independent of the presence of the saturating fluid:

$$K_{sat} = K_{dry} + \frac{K_f \alpha^2}{\varphi + \frac{K_f}{K_m}\left(\alpha - \varphi\right)} \quad (1)$$

$$G_{sat} = G_{dry} \quad (2)$$

Where α is the Biot-Willis coefficient [4]:

$$\alpha = 1 - \frac{K_{dry}}{K_m} \quad (3)$$

The moduli are related to the seismic velocities and density by:

$$K = \rho\left(V_p^2 - \frac{4}{3}V_s^2\right) \quad (4)$$

$$G = \rho V_s^2 \quad (5)$$

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Berryman [5] gave a concise derivation of Gassmann equations for an isotropic and homogeneous medium using the quasi-static poroelastic theory. Other forms of Equation (1) can be found in Mavko et al. [6]. Zimmerman [7] presented an equivalent form in terms of compressibility. However, Equation (1) is probably the most intuitive in describing the effect of fluid presence on the bulk modulus.

White and Castagna [8] argued that, since all input parameters for Gassmann equations carry some degrees of uncertainty, a fluid modulus inversion should be performed using a probabilistic approach. Artola and Alvarado [9] evaluated the effect of uncertainty of different input parameters and showed that the computed compressional velocity of a saturated rock is most sensitive to uncertainties in the rock bulk density and the dry bulk and shear moduli, while other parameters (porosity, the grain matrix and fluid bulk moduli) have negligible effects.

Note that the three parameters: dry frame modulus (K_{dry}), Biot-Willis coefficient (α), and grain matrix bulk modulus (K_m) are related by Equation (3); in many instances they are unknown. The fluid saturated bulk modulus (K_{sat}) and fluid bulk modulus (K_f) can also be unknown (e.g. in fluid substitution problem). As a result, ad-hoc and empirical correlations have been proposed to address this problem. There are many instances Biot-Willis coefficient is assumed to be 1 due to the lack of a better estimate. For sandstone at high differential pressure (40MPa), Han and Batzle [10] proposed α to be a polynomial function of porosity:

$$\alpha = 3.206\varphi - 3.349\varphi^2 + 1.143\varphi^3 \quad (6)$$

In this study, we present 3 new equivalent forms of Gassmann equation that are useful for different scenarios of available data. We apply these equations to several sets of laboratory measurements. A stochastic simulation is performed to examine the effect of uncertainty and/or measurement errors on calculated grain matrix bulk modulus and Biot-Willis coefficient.

2. The new equivalent Gassmann equations

2.1. When (K_{dry}, K_{sat}, K_f and φ) are known

This is generally the case for laboratory measurements on dry and wet rock samples (e.g. dry and brine saturated acoustic velocities are measured as functions of differential pressure along with rock porosity). In this case we can rewrite Equation (1) as a function of Biot-Willis coefficient α (see Appendix A for a detailed derivation):

$$\alpha^2 - (\varphi+1)\left(1-\frac{K_{dry}}{K_{sat}}\right)\alpha + \varphi\left(1-\frac{K_{dry}}{K_{sat}}\right)\left(1-\frac{K_{dry}}{K_f}\right) = 0 \quad (7)$$

Equation (7) is a quadratic equation $A\alpha^2 + B\alpha + C = 0$, where all coefficients can be readily calculated.

$$A = 1 \quad (8)$$

$$B = -(\varphi+1)\left(1-\frac{K_{dry}}{K_{sat}}\right) \quad (9)$$

$$C = \varphi\left(1-\frac{K_{dry}}{K_{sat}}\right)\left(1-\frac{K_{dry}}{K_f}\right) \quad (10)$$

This simple quadratic equation has two solutions:

$$\alpha_{1,2} = \frac{-B \pm \sqrt{\Delta}}{2A}, \text{where } \Delta = B^2 - 4AC \quad (11)$$

However, Berryman and Milton [11] showed that α is physically bounded between 0 and 1. Equations (9) and (10) show that B is negative since $K_{dry} < K_{sat}$, and C is also negative since $K_f < K_{dry}$ for consolidated rocks. This means $\Delta = B^2 - 4AC > B^2$ and thus, $\alpha_1 = \frac{-B+\sqrt{\Delta}}{2A}$ is the only possible solution since α_2 is negative.

The corresponding grain matrix bulk modulus then can be calculated from Equation (3):

$$K_m = \frac{K_{dry}}{1-\alpha} \quad (12)$$

Therefore, instead of having two non-linear equations for two unknowns (α and K_m), we reduce the problem to one simple quadratic equation, Equation (7), that always gives one physically realistic solution.

This provides an independent methodology to calculate the grain matrix bulk modulus of a rock from SCAL laboratory acoustic measurements of dry and saturated rock samples. Traditionally, the grain matrix bulk moduli are estimated from averages of the rock mineralogical composition (e.g. Voigt-Reuss-Hill average or Hashin-Shtrikman bounds). These bounds may carry large uncertainties since many minerals, especially clays, have a high variance in their bulk modulus values depending on the measurement conditions [12, 13]. We can further postulate that: (a) the grain matrix calculated from Gassmann equation (using Equations (8) to (12)) must lie between the two bounds obtained from mixture theory, and (b) the calculated grain matrix values are insensitive to the first order to the applied pressure. Equations (7) to (11) can also be used to verify the applicability of existing empirical or ad-hoc correlations (such as Equation (6) to estimate Biot-Willis coefficient) for different rocks.

2.2. When (K_{sat1}, K_{sat2}, K_{f1}, K_{f2} and φ) are known

This case can be encountered in the field. The same rock can be fully saturated with brine in one well while having oil or gas in another well; or it can have varying saturations in the same well. Acoustic logs, and density - porosity logs are available. In this case, K_{dry}, K_m, and α are unknown in a system of three non-linear equations (two Equations (1) for two different saturation fluids and Equation (3)). Starting from Equation (7) instead, we end up with (see Appendix B for detailed derivations):

$$\varphi\left(\frac{1}{K_{sat1}K_{f1}} - \frac{1}{K_{sat2}K_{f2}}\right)K_{dry} = \varphi\left(\frac{1}{K_{f1}} - \frac{1}{K_{f2}}\right) - [\alpha(\varphi+1)-\varphi]\left(\frac{1}{K_{sat1}} - \frac{1}{K_{sat2}}\right) \quad (13)$$

We can write Equation (13) in a more convenient form for numerical calculations:

$$\varphi\left(\frac{K_{sat2}}{K_{f1}} - \frac{K_{sat1}}{K_{f2}}\right)K_{dry} = \varphi\left(\frac{1}{K_{f1}} - \frac{1}{K_{f2}}\right)K_{sat1}K_{sat2} + (K_{sat1} - K_{sat2})[\alpha(\varphi+1)-\varphi] \quad (14)$$

K_{dry}, α, and K_m can now be calculated very quickly using a simple iteration using Equation (14) and Equation (7) as follows:

- Step 1: Make an initial guess, for example:

$$K_{dry} = 0.5 \times min\{K_{sat1}, K_{sat2}\}$$

- Step 2: Use guessed K_{dry} value in Equation (7) to find two Biot-Willis coefficients α_{f1} and α_{f2} (for two saturations):

$$\alpha^2 - (\varphi + 1)\left(1 - \frac{K_{dry}}{K_{sat1}}\right)\alpha + \varphi\left(1 - \frac{K_{dry}}{K_{sat1}}\right)\left(1 - \frac{K_{dry}}{K_{f1}}\right) = 0$$

$$\alpha^2 - (\varphi + 1)\left(1 - \frac{K_{dry}}{K_{sat2}}\right)\alpha + \varphi\left(1 - \frac{K_{dry}}{K_{sat2}}\right)\left(1 - \frac{K_{dry}}{K_{f2}}\right) = 0$$

- Step 3: Take the average for a new Biot-Willis coefficient:

$$\alpha = (\alpha_{f1} + \alpha_{f2})/2$$

- Step 4: Use this new α in Equation (14) to find new K_{dry}.

- Step 5: Repeat steps 2 to 4 until K_{dry} converges:

$$\left|\frac{K_{dry,new} - K_{dry,old}}{K_{dry,new}}\right| < \varepsilon$$

- Step 6: Use Equations (7) and (12) to find corresponding α and K_m.

Note that we have assumed there are no softening or hardening effects caused by the saturating fluids on the grain bulk modulus (K_m is constant). The second assumption is that the rock dry frame is stiffer than both fluids, $K_{dry} > max\{K_{f1}, K_{f2}\}$, so that Equation (7) still gives only one positive (physically realistic) root. This assumption is generally valid for consolidated sedimentary rocks.

2.3. When (K_m, K_{sat}, K_f and φ) are known

In this case K_{dry} and α are unknown. An instance for this case is that fluid data, fluid saturation, acoustic log (V_p, V_s) and density-porosity log are available while K_m is estimated from the mineralogical composition of the rock (FTIR, XRD, thin section of rock cuttings, or mineralogy log). The Biot-Willis coefficient can be estimated directly from the following equivalent Gassmann equation (see Appendix C for detailed derivations):

$$\left[\varphi(K_m - K_f) - K_f\left(1 - \frac{K_{sat}}{K_m}\right)\right]\alpha = \varphi\left[(K_m - K_f) - K_{sat}\left(1 - \frac{K_f}{K_m}\right)\right] \quad (15)$$

And applying this α to Equation (3) gives K_{dry}. This is equivalent to the K_{dry} solution of Zhu and McMechan [14].

3. Numerical applications

3.1. Han and Batzle's sandstone data

We applied Equation (7) to the pressure dependent dry and brine saturated velocities and moduli of a porous sandstone sample published by Han and Batzle [10] (Figure 1). The wet and dry densities are back-calculated from Equations (4) and (5). The porosity values are calculated using the density relationship:

$$\rho_{sat} = \rho_{dry} + \varphi\rho_f \quad (16)$$

The calculated Biot-Willis coefficient and grain matrix modulus as functions of pressure are plotted in Figure 1. The relatively constant value of the grain bulk modulus (39GPa) as a function of pressure is a good indicator that Gassmann equation is applicable for this rock. The variation of grain bulk modulus at low confining pressure (< 10MPa) is possibly due to higher uncertainty in input values (i.e. higher noise-to-signal ratio from velocity signals).

The Biot-Willis coefficient profile is remarkably similar to the result measured on a 26% porosity Boise sandstone sample by Fatt [15]. Note that Gassmann equation gave a higher value (0.73 at 40MPa) than Han and Batzle's Equation (6) (0.63).

3.2. Coyner's limestone data

We employed the iteration procedure using Equations (7) to (14) on water and benzene saturated Bedford limestone sample published by Coyner [16] (Figure 2). In his experiment at room temperature, the fluid pore pressure in both saturation cases was maintained at 10MPa. The porosity of the rock is 11.9%. The shear modulus profile is almost identical for all vacuum dry, water saturated, and benzene saturated cases, suggesting that Gassmann equation is valid for this rock. At 10MPa pore pressure, K_{water} = 2.24GPa, and $K_{benzene}$ = 1.21GPa.

The back-calculated dry bulk modulus is also plotted (as a dashed line) against the various measured moduli in Figure 2. The profile is consistently higher than the measured vacuum dry bulk modulus profile by approximately 2.5GPa (or 5 - 9%). This is another evidence supporting the argument that the vacuum dry measured bulk modulus is too dry and should not

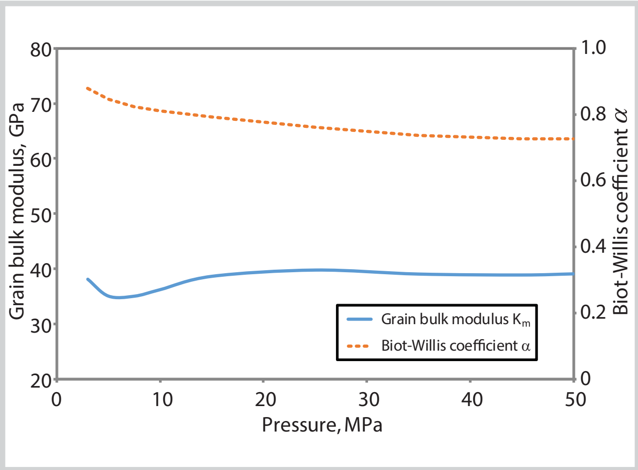

Figure 1. *Grain bulk modulus and Biot-Willis coefficient of a sandstone sample as a function of pressure calculated from its dry and brine saturated moduli [10] using Gassmann equation.*

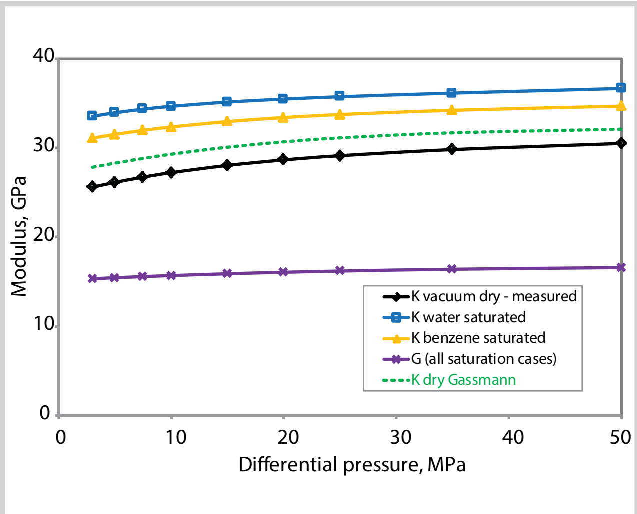

Figure 2. *Bulk and shear moduli as a function of differential pressure for Bedford limestone [6]. The dashed line is the dry bulk modulus calculated from Gassmann equation, consistently higher than the vacuum dry measured data.*

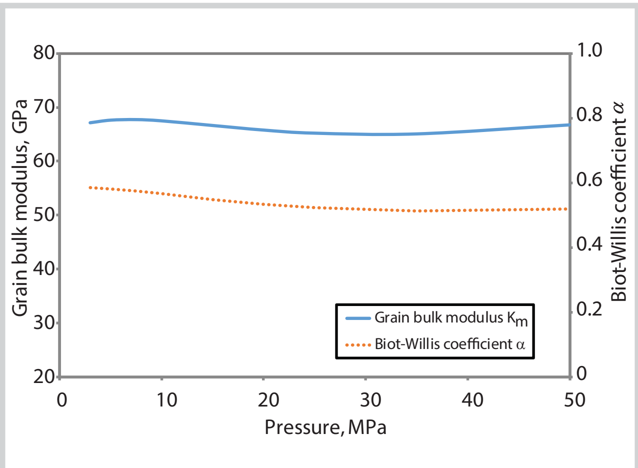

Figure 3. *Calculated grain matrix bulk modulus and Biot-Willis coefficient of Bedford limestone sample as a function of pressure using Gassmann equation from its water - and benzene - saturated moduli [16].*

be used in Gassmann equation [6, 17]. We could have applied the measured vacuum dry and either water- or benzene-saturated moduli values on Equation (7), but that approach would give unrealistically high grain matrix bulk modulus.

The grain matrix bulk modulus and Biot-Willis coefficient profiles obtained from the rock water- and benzene-saturated moduli are shown in Figure 3. While the grain matrix bulk modulus is similar to Coyner's reported value of 65GPa, the back-calculated Biot-Willis coefficient profile decreases from 0.6 to 0.53, significantly lower than the commonly assumed value of 1 while significantly higher than the estimated value of 0.34 obtained from Han and Batzle's 2004 correlation.

3.3. Effects of input data errors on calculated grain bulk modulus

Measured values always have some associated errors. Velocities, especially shear wave velocities may carry significant uncertainties. We would like to determine the effects of uncertainties from porosity, K_{dry}, K_{sat}, and K_f to the uncertainty of the predicted K_m. Since the relationship in Equation (7) is not linear, a Monte Carlo (stochastic) simulation was used.

Table 1 summarises the input parameter values [18] and their estimated ranges of uncertainties. The rock sample is a Berea sandstone sample with Voigt-Reuss-Hill average grain bulk modulus of 39.6GPa from its mineralogical composition. All parameters were assumed to have a normal distribution with means being the measured values and the errors represent the 95% confidence interval. Thus, the relative error (uncertainty) of each parameter is defined as:

$$\%\ error = \frac{2s}{mean} \times 100\% \qquad (17)$$

Where s is the standard deviation of the parameter's sample.

For each set of perturbed errors, 10,000 sets of (porosity, dry bulk modulus, wet bulk modulus, and fluid modulus) values were generated to compute 10,000 grain bulk moduli, which are then analysed for the mean value and standard deviation.

In our base case, porosity is assigned a 1% error, K_{dry} and K_{sat} are each assigned a 3% error, and K_f carries a 10% uncertainty. The resulting K_m is also a Gaussian distribution with a mean of 44.6GPa and a standard deviation of 3.45GPa. The 95% confidence interval is,

Table 1. Mean (measured) values of a Berea sandstone sample [18] and ranges of uncertainties used in Monte Carlo simulations

Parameters	Mean values (measured)	% error	Standard deviation
Porosity	17.6%	±1 - 15%	±0.09 - 1.3%
Effective dry bulk modulus	16.8GPa	±1 - 15%	±0.25 - 1.26GPa
Effective wet bulk modulus	21.1GPa	±1 - 15%	±0.32 - 1.58GPa
Fluid bulk modulus (water)	2.2GPa	±0 - 30%	±0 - 0.33GPa

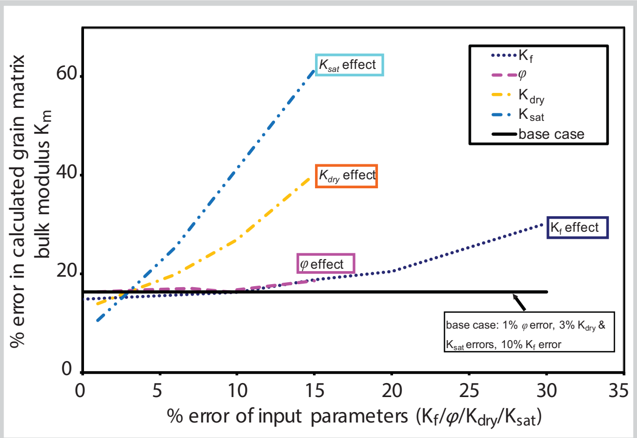

Figure 4. The uncertainty of the computed grain matrix bulk modulus K_m using Gassmann equation as functions of percent error in one input parameter (K_f, α, K_{dry} or K_{sat}), while the remaining input parameters carry the same errors as the base case. Errors from K_{sat} and K_{dry} have the largest impacts on the uncertainty of calculated K_m. Porosity and fluid bulk modulus, on the other hand, show negligible effects.

therefore, from 37.7GPa to 51.5GPa (or 16% error). The Biot-Willis coefficient α is also a Gaussian distribution with a mean of 0.62 and a standard deviation of 0.03. The 95% confidence interval is from 0.56 to 0.68 (or 10% error).

Figure 4 shows the uncertainty of the computed grain matrix bulk modulus K_m as functions of percent error in one input parameter (K_f, φ, K_{dry}, or K_{sat}), while the remaining input parameters carry the same uncertainties as of the base case. Errors from K_{sat} and K_{dry} have the largest effects on the uncertainty of K_m. Minus errors in K_{dry} and K_{sat} (even within laboratory measurement standard) can result in a large error in the estimated value of K_m. Porosity and fluid bulk modulus, on the other hand, show negligible influence. This result is not surprising, as K_f, φ and K_m should be uncorrelated parameters.

4. Conclusions

Three equivalent forms of Gassmann equation were presented that can be useful for the determination of Biot-Willis coefficient, dry bulk modulus, and/or grain matrix bulk modulus of a rock. We demonstrated the applicability of these equations using several sets of published laboratory measurements and the implications of the results for other estimations of rock properties. A stochastic simulation was also performed to examine the effect of uncertainty and/or measurement errors on calculated grain matrix bulk modulus. The results showed that the calculated grain matrix bulk modulus is relatively constant with applied differential pressure (up to 50MPa) for sedimentary rocks. However, the estimation is very sensitive to the uncertainty of dry and saturated bulk modulus values. Our new forms of Gassmann equation can be used to effectively quantify the uncertainty of dry and saturated bulk modulus (and subsequently, the seismic velocities) in fluid identification, fluid substitution, or reservoir monitoring applications.

NOMENCLATURE

K: bulk modulus (GPa or psi)

K_{sat}: saturated bulk modulus (GPa or psi)

K_{dry}: dry (frame) bulk modulus (GPa or psi)

K_m: grain (matrix) bulk modulus (GPa or psi)

K_f: fluid bulk modulus (GPa or psi)

G: shear modulus (GPa or psi)

G_{sat}: saturated shear modulus (GPa or psi)

G_{dry}: dry (frame) shear modulus (GPa or psi)

α: Biot-Willis coefficient (dimensionless)

φ: porosity (dimensionless)

ρ: density (g/cc)

V_p: compressional wave velocity (km/s)

V_s: shear wave velocity (km/s)

APPENDIX A: Derivation of Equation (7)

From Equation (3) we can write:

$$\frac{K_f}{K_m} = (1-\alpha)\frac{K_f}{K_{dry}} \tag{A.1}$$

Rewriting Equation (1) as a function of α gives:

$$(K_{sat}-K_{dry})\left[\varphi+(1-\alpha)\frac{K_f}{K_{dry}}(\alpha-\varphi)\right] = K_f\alpha^2 \tag{A.2}$$

$$\alpha^2\left[K_f+\frac{K_f}{K_{dry}}(K_{sat}-K_{dry})\right]-(\varphi+1)\frac{K_f}{K_{dry}}(K_{sat}-K_{dry})\alpha + \varphi(K_{sat}-K_{dry})\left(\frac{K_f}{K_{dry}}-1\right)=0 \quad \text{(A.3)}$$

$$\frac{K_f K_{sat}}{K_{dry}}\alpha^2-\left[(\varphi+1)\frac{K_f}{K_{dry}}(K_{sat}-K_{dry})\right]\alpha + \varphi(K_{sat}-K_{dry})\left(\frac{K_f}{K_{dry}}-1\right)=0 \quad \text{(A.4)}$$

Multiplying both sides with $\frac{K_{dry}}{K_f K_{sat}}$ gives

$$\alpha^2-(\varphi+1)\left(1-\frac{K_{dry}}{K_{sat}}\right)\alpha+\varphi\left(1-\frac{K_{dry}}{K_{sat}}\right)\left(1-\frac{K_{dry}}{K_f}\right)=0,$$

which is Equation (7).

APPENDIX B: Derivation of Equation (13)

If the same rock is subjected to two different saturation fluids, then we have two equations in the form of Equation (7):

$$\alpha^2-(\varphi+1)\left(1-\frac{K_{dry}}{K_{sat1}}\right)\alpha+\varphi\left(1-\frac{K_{dry}}{K_{sat1}}\right)\left(1-\frac{K_{dry}}{K_{f1}}\right)=0 \quad \text{(B.1)}$$

$$\alpha^2-(\varphi+1)\left(1-\frac{K_{dry}}{K_{sat2}}\right)\alpha+\varphi\left(1-\frac{K_{dry}}{K_{sat2}}\right)\left(1-\frac{K_{dry}}{K_{f2}}\right)=0 \quad \text{(B.2)}$$

Subtracting Equation (B.2) from (B.1) gives:

$$(\varphi+1)\left(\frac{1}{K_{sat1}}-\frac{1}{K_{sat2}}\right)K_{dry}\alpha=\varphi K_{dry}\left[\left(\frac{1}{K_{f1}}-\frac{1}{K_{f2}}\right) + \left(\frac{1}{K_{sat1}}-\frac{1}{K_{sat2}}\right)-K_{dry}\left(\frac{1}{K_{sat1}K_{f1}}-\frac{1}{K_{sat2}K_{f2}}\right)\right] \quad \text{(B.3)}$$

Canceling K_{dry} both sides and rearranging Equation (B.3) leads to:

$$\varphi\left(\frac{1}{K_{sat1}K_{f1}}-\frac{1}{K_{sat2}K_{f2}}\right)K_{dry}=\varphi\left(\frac{1}{K_{f1}}-\frac{1}{K_{f2}}\right)-\left[\alpha(\varphi+1)-\varphi\right]\left(\frac{1}{K_{sat1}}-\frac{1}{K_{sat2}}\right)$$

which is Equation (13). Equation (14) then can be readily obtained by multiplying both sides by ($K_{sat1}\times K_{sat2}$).

APPENDIX C: Derivation of Equation (15)

If K_m value can be obtained (e.g. using mixture theory), then one can substitute $K_{dry}=(1-\alpha)K_m$ into Gassmann equation and rearrange Equation (1) as a function of α, the Biot-Willis coefficient only:

$$\left[K_{sat}-(1-\alpha)K_m\right]\left[\varphi+\frac{K_f}{K_m}(\alpha-\varphi)\right]=K_f\alpha^2 \quad \text{(C.1)}$$

Expanding LHS and subtracting $K_f\alpha^2$ both sides, we have:

$$\alpha\left[\frac{K_{sat}K_f}{K_m}+K_m\varphi-(\varphi+1)K_f\right]+\varphi\left[(K_{sat}-K_m)+K_f-\frac{K_fK_{sat}}{K_m}\right]=0 \quad \text{(C.2)}$$

or equivalently,

$$\left[\varphi(K_m-K_f)-K_f\left(1-\frac{K_{sat}}{K_m}\right)\right]\alpha=\varphi\left[(K_m-K_f)-K_{sat}\left(1-\frac{K_f}{K_m}\right)\right]$$

which is Equation (15).

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